RESEARCH IN BIOLOGY USING COMPUTATION

APPLICATIONS OF SOME NEW TRANSMUTED CUMULATIVE DISTRIBUTION FUNCTIONS IN POPULATION DYNAMICS

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Conflict of Interest

None declared.

Vesselin Kyurkchiev¹, Anton Iliev¹,²*, Nikolay Kyurkchiev¹
¹Faculty of Mathematics and Informatics, University of Plovdiv Paisii Hilendarski, 24, Tzar Asen Str., 4000 Plovdiv, Bulgaria
²Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. G. Bonchev Str., Bl. 8, 1113 Sofia, Bulgaria
*To whom correspondence should be addressed.

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Abstract

Motivation: In literature, several transformations exists to obtain a new cumulative distribution function (cdf) using other(s) well-known cdf(s).

Results: In this note we find applications of some new cumulative distribution function transformations to construct a family of sigmoidal functions based on the Verhulst logistic function. We prove estimates for the Hausdorff approximation of the shifted Heaviside step function by means of this family. Numerical examples, illustrating our results are given.

Keywords: Cumulative Distribution Function, Logistic Function, Shifted Heaviside Step Function, Hausdorff Distance, Upper and Lower Bounds.

Contact: aii@uni-plovdiv.bg

1 Introduction

In literature, several transformations exists to obtain a new cumulative distribution function (cdf) using other(s) well-known cdf(s) (Aryal & Tsokos, 2009; Aryal, 2013; Gupta, R. G., Gupta, P. L., & Gupta, R. D., 1998; Khan & King, 2013; Kumar, Singh, & Singh, 2015a; Kumar, Singh & Singh, 2015b; Kumar, Singh & Singh, 2017).

Definition 1 Another popular transformation by using a (cdf) F(t) is (Kumar, Singh, & Singh, 2015a):

\[ G(t) = \frac{1}{e^{a+1}} (e^{F(t)} - 1) \]  
(1)

The transformation (1) has great applications in data analysis.

Definition 2 Define the logistic (Verhulst) function \( f \) on \( \mathbb{R} \) as

\[ f(t) = \frac{1}{1+e^{-k_0 t}} \]  
(2)

The logistic function belongs to the important class of smooth sigmoidal functions arising from population and cell growth models.

Since then the logistic function finds applications in many scientific fields, including biology, population dynamics, chemistry, demography, economics, geoscience, mathematical psychology, probability, financial mathematics, statistics, insurance mathematics to name a few (Anguelov & Markov, 2016; Lente, 2015; Kyurkchiev & Markov, 2016a; Kyurkchiev, 2016a; Costarelli & Spigler, 2013; Kyurkchiev & Markov, 2014; Kyurkchiev & Markov, 2016; Kyurkchiev & Markov, 2016b).

Definition 3 The (interval) step function is:

\[ h_{t_0}(t) = \begin{cases} 0, & \text{if } t < t_0, \\ [0,1], & \text{if } t = t_0, \\ 1, & \text{if } t > t_0. \end{cases} \]
usually known as shifted Heaviside step function.

**Definition 4** (Hausdorff, 2005; Sendov, 1990) The Hausdorff distance (the H-distance) \( \rho(f, g) \) between two interval functions \( f, g \) on \( \Omega \subseteq \mathbb{R} \), is the distance between their completed graphs \( F(f) \) and \( F(g) \) considered as closed subsets of \( \Omega \times \mathbb{R} \). More precisely,

\[
\rho(f, g) = \max \{ \sup_{B \in F(f)} \inf_{A \in F(g)} ||A - B||, \sup_{B \in F(g)} \inf_{A \in F(f)} ||A - B|| \},
\]

wherein \( ||.|| \) is any norm in \( \mathbb{R}^2 \), e.g. the maximum norm \( \|\|(t, x)\|\| = \max \{|t|, |x|\} \); hence the distance between the points \( A = (t_A, x_A) \), \( B = (t_B, x_B) \) in \( \mathbb{R}^2 \) is \( \|A - B\| = \max \{|t_A - t_B|, |x_A - x_B|\} \).

In this paper we discuss several computational, modelling and approximation issues related to two familiar classes of sigmoidal functions—these are the families of transmuted cumulative distribution functions.

## 2 Methods

1. Let us consider the following sigmoid

\[
G(t) = \frac{1}{e-1}(e^{f(t)} - 1)
\]

with

\[
G(t_0) = \frac{1}{2}, \quad t_0 = \frac{1}{k} \ln \frac{1}{\ln(2) - 1}
\]

based on (1) with the Verhulst logistic function \( f(t) \).

The one–sided H-distance \( d = \rho(h_{t_0}, G) \) between the shifted Heaviside step function \( h_{t_0} \) and the sigmoidal function \( G \) satisfies the relation:

\[
G(t_0 + d) = \frac{1}{e-1}(e^{f(t_0+d)} - 1) = 1 - d.
\]

The following theorem gives upper and lower bounds for \( d = d(k) \)

**Theorem 2.1** The one–sided H-distance \( d(k) \) between the function \( h_{t_0} \) and the function \( G \) can be expressed in terms of the rate parameter \( k \) for any real \( k \geq 2 \) as follows:

\[
d_l = \frac{1}{2.5(1+0.254884k)} < d < \frac{\ln 2.5(1+0.254884)}{2.5(1+0.254884k)} = d_r.
\]

**Proof.** We define the functions

\[
F_1(d) = \frac{1}{e-1}(e^{f(t_0+d)} - 1) - 1 + d
\]

\[
G_1(d) = -\frac{1}{2} + (1 + 0.254884k)d.
\]

From Taylor expansion

\[
\frac{1}{e-1}(e^{f(t_0+d)} - 1) - 1 + d - (-\frac{1}{2} + (1 + 0.254884k)d)
\]

we see that the function \( G_1(d) \) approximates \( F_1(d) \) with \( d \to 0 \) as \( O(d^2) \) (see Figure 1).

In addition \( G_1'(d) > 0 \) and for \( k \geq 2 \)

\[
G_1(d_1) < 0; \quad G_1(d_2) > 0.
\]

This completes the proof of the inequalities (7).

![Fig. 1 - The functions \( F_1(d) \) and \( G_1(d) \) for \( k = 20 \).](image)

The generated sigmoidal function \( G(t) \) for \( k = 20 \) is visualized on Figure 2

![Fig. 2 - The H-distance \( d(k) \) between the functions \( h_{t_0} \) and \( G \) for \( k = 20 \) is \( d = 0.105561 \); \( d_l = 0.0655987 \); \( d_r = 0.178704 \).](image)

Some computational examples using relations (7) are presented in Table 1. The third column of Table 1 contains the value of \( d \) for prescribed values of \( k \) computed by solving the nonlinear equation (6).

<table>
<thead>
<tr>
<th>( k )</th>
<th>( d_l )</th>
<th>( d ) computed by (6)</th>
<th>( d_r )</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>0.0462014</td>
<td>0.0747728</td>
<td>0.142182</td>
</tr>
<tr>
<td>40</td>
<td>0.0357291</td>
<td>0.0627923</td>
<td>0.119042</td>
</tr>
<tr>
<td>50</td>
<td>0.0291032</td>
<td>0.0541761</td>
<td>0.102935</td>
</tr>
<tr>
<td>100</td>
<td>0.015101</td>
<td>0.0296749</td>
<td>0.0633183</td>
</tr>
<tr>
<td>500</td>
<td>0.00311425</td>
<td>0.00617859</td>
<td>0.0179747</td>
</tr>
<tr>
<td>1000</td>
<td>0.00156321</td>
<td>0.0048109</td>
<td>0.0100999</td>
</tr>
</tbody>
</table>

**Definition 5** Another popular transformation by using a (cdf) \( F(t) \) is (Kumar, Singh & Singh, 2017):
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\[ G_1(t) = e^{\frac{1}{f(t)}}. \] (10)

2. Let us consider the following sigmoid
\[ G_1(t) = e^{\frac{1}{f(t)}} \] (11)
with
\[ G_1(t_0) = \frac{1}{2}, t_0 = -\frac{1}{k}\ln(\ln2) \] (12)
based on (10) with the Verhulst logistic function \( f(t) \).

The one-sided H-distance \( d_1 = \rho(h_{t_0}, G_1) \) between the shifted Heaviside step function \( h_{t_0} \) and the sigmoidal function \( G_1 \) satisfies the relation:
\[ G_1(t_0 + d_1) = e^{\frac{1}{f(t_0 + d_1)}} = 1 - d_1, \] (13)
The following theorem gives upper and lower bounds for \( d_1 = d_1(k) \).

**Theorem 2.2** The one-sided H-distance \( d_1(k) \) between the function \( h_{t_0} \) and the function \( G_1 \) can be expressed in terms of the rate parameter \( k \) for any real \( k \geq 2 \) as follows:
\[ d_1 = \frac{1}{2.5(1+0.346574k)} < d_1 < \frac{\ln2.5(1+0.346574k)}{2.5(1+0.346574k)} = d_{r_1}. \] (14)

The proof follows the ideas given in this paper and will be omitted.

**Fig. 3** - The H-distance \( d_1(k) \) between the functions \( h_{t_0} \) and \( G_1 \) for \( k = 30 \) is \( d_1 = 0.0735233 \).

**Fig. 4** - Comparison between \( G \) (red) and \( G_1 \) (green) for \( k = 20 \).

3 **Results**
To achieve our goal, we obtain new estimates for the one-sided H-distance between a shifted Heaviside step function and its best approximating family of transmuted cumulative distribution functions—these are the families of functions \( G(t) \) and \( G_1(t) \) based on the Verhulst logistic function.

Numerical examples, illustrating our results are given. In some cases the approximation of shifted Heaviside function by \( G_1(t) \) is better in comparison to its approximation by \( G(t) \) (see Figure 4).


**Fig. 5** - Software tools in CAS Mathematica.
We propose a software module within the programming environment CAS Mathematica for the analysis of the considered families of transmuted cumulative distribution functions. The module offers the following possibilities:
- generation of the functions \( G(t) \) and \( G_1(t) \) under user defined values of the reaction rate \( k \) and \( r \);
- calculation of the H-distance between the Heaviside function \( h_G \) and the sigmoidal functions \( G(t) \) and \( G_1(t) \);
- software tools for animation and visualization.

4 Appendix

Focusing on the shifted logistic function

\[
 f_r(t) = \frac{1}{1 + e^{-(t-r)}} \tag{15}
\]

and the shifted function

\[
 G_r(t) = \frac{1}{e-1} \left( e^{f_r(t)} - 1 \right). \tag{16}
\]

We examine the following experimental data (biomass) for \textit{Xantobacter autotrophycum} by the model \( G_r(t) \).

Table 2. The experimental data (biomass) for \textit{Xantobacter autotrophycum} and approximation by \( G_r(t) \) for \( k = 20 \) and \( r = 0.1 \)

<table>
<thead>
<tr>
<th>( t )</th>
<th>Biomass</th>
<th>( G_r(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.104</td>
<td>0.0736774</td>
</tr>
<tr>
<td>0.065</td>
<td>0.233</td>
<td>0.229003</td>
</tr>
<tr>
<td>0.099</td>
<td>0.39</td>
<td>0.372755</td>
</tr>
<tr>
<td>0.125</td>
<td>0.507</td>
<td>0.50254</td>
</tr>
<tr>
<td>0.145</td>
<td>0.618</td>
<td>0.602883</td>
</tr>
<tr>
<td>0.188</td>
<td>0.766</td>
<td>0.784021</td>
</tr>
<tr>
<td>0.233</td>
<td>0.88</td>
<td>0.899886</td>
</tr>
</tbody>
</table>

From Figure 6 it can be seen that the results are satisfactory. We point out that in similar "exponential" data type the results are near to Gompertz growth model.

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References


Applications of some new transmuted cumulative distribution functions in population dynamics


ПРИМЕНЕНИЕ НЕКОТОРЫХ НОВЫХ ТРАНСМУТИРОВАННЫХ КУМУЛЯТИВНЫХ ФУНКЦИЙ РАСПРЕДЕЛЕНИЯ В ДИНАМИКЕ ПОПУЛЯЦИЙ

Финансирование
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Весселин Кюркчиев (Vesselin Kyurkchiev)1, Антон Илиев (Anton Iliev)1,2*, Николай Кюркчиев (Nikolay Kyurkchiev)1
1Факультет математики и информатики, Пловдивский университет «Паисий Хилендарски» (University of Plovdiv Paisii Hilendarski), ул. Царь Асен 24, 4000 Пловдив, Болгария
2Институт математики и информатики, Болгарская академия наук, Ул. Акад. Г. Бончева, д. 8, 1113 София, Болгария
*Корреспондирующий автор.

Редактор: Джанкарло Кастельяно

Аннотация
Мотивация: В литературе представлено несколько преобразований для получения новой кумулятивной функции распределения (cdf) с помощью другой(их) известной(ых) cdf.
Результаты: В данной работе найдено применение некоторым новым преобразованиям кумулятивной функции распределения для построения семейства симметричных функций на основе логистической функции Ферхюльста.
Посредством этого семейства мы доказываем предположения для хаусдорфовой аппроксимации сдвинутой ступенчатой функции Хевисайда. Приводятся численные примеры, иллюстрирующие полученные результаты.

Ключевые слова: Кумулятивная функция распределения, логистическая функция, сдвинутая ступенчатая функция Хевисайда, хаусдорфово расстояние, верхний и нижний пределы.