

## RESEARCH IN BIOLOGY USING COMPUTATION

### APPLICATIONS OF SOME NEW TRANSMUTED CUMULATIVE DISTRIBUTION FUNCTIONS IN POPULATION DYNAMICS

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#### **Conflict of Interest**

*None declared.*

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#### **Abstract**

**Motivation:** In literature, several transformations exists to obtain a new cumulative distribution function (cdf) using other(s) well-known cdf(s).

**Results:** In this note we find applications of some new cumulative distribution function transformations to construct a family of sigmoidal functions based on the Verhulst logistic function.

We prove estimates for the Hausdorff approximation of the shifted Heaviside step function by means of this family. Numerical examples, illustrating our results are given.

**Keywords:** Cumulative Distribution Function, Logistic Function, Shifted Heaviside Step Function, Hausdorff Distance, Upper and Lower Bounds.

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#### **1 Introduction**

In literature, several transformations exists to obtain a new cumulative distribution function (cdf) using other(s) well-known cdf(s) (Aryal & Tsokos, 2009; Aryal, 2013; Gupta, R. G., Gupta, P. L. & Gupta, R. D., 1998; Khan & King, 2013; Kumar, Singh, & Singh, 2015a; Kumar, Singh & Singh, 2015b; Kumar, Singh & Singh 2017).

**Definition 1** Another popular transformation by using a (cdf)  $F(t)$  is (Kumar, Singh, & Singh, 2015a):

$$G(t) = \frac{1}{e-1}(e^{F(t)} - 1) \quad (1)$$

The transformation (1) has great applications in data analysis.

**Definition 2** Define the logistic (Verhulst) function  $f$  on  $\mathbb{R}$  as

$$f(t) = \frac{1}{1+e^{-kt}}. \quad (2)$$

The logistic function belongs to the important class of smooth sigmoidal functions arising from population and cell growth models.

Since then the logistic function finds applications in many scientific fields, including biology, population dynamics, chemistry, demography, economics, geoscience, mathematical psychology, probability, financial mathematics, statistics, insurance mathematics to name a few (Anguelov & Markov, 2016; Lente, 2015; Kyurkchiev & Markov, 2016a; Kyurkchiev, 2016a; Costarelli & Spigler, 2013; Kyurkchiev & Markov, 2014; Kyurkchiev & Markov, 2015; Kyurkchiev & Markov, 2016b).

**Definition 3** The (interval) step function is:

$$h_{t_0}(t) = \begin{cases} 0, & \text{if } t < t_0, \\ [0,1], & \text{if } t = t_0, \\ 1, & \text{if } t > t_0, \end{cases}$$

usually known as *shifted Heaviside step function*.

**Definition 4** (Hausdorff, 2005; Sendov, 1990) The Hausdorff distance (the H-distance)  $\rho(f, g)$  between two interval functions  $f, g$  on  $\Omega \subseteq \mathbb{R}$  is the distance between their completed graphs  $F(f)$  and  $F(g)$  considered as closed subsets of  $\Omega \times \mathbb{R}$ . More precisely,

$$\rho(f, g) = \max\left\{ \sup_{A \in F(f)} \inf_{B \in F(g)} \|A - B\|, \sup_{B \in F(g)} \inf_{A \in F(f)} \|A - B\| \right\}, \quad (3)$$

wherein  $\|\cdot\|$  is any norm in  $\mathbb{R}^2$ , e. g. the maximum norm  $\|(t, x)\| = \max\{|t|, |x|\}$ ; hence the distance between the points  $A = (t_A, x_A)$ ,  $B = (t_B, x_B)$  in  $\mathbb{R}^2$  is  $\|A - B\| = \max(|t_A - t_B|, |x_A - x_B|)$ .

In this paper we discuss several computational, modelling and approximation issues related to two familiar classes of sigmoidal functions—these are the families of transmuted cumulative distribution functions.

## 2 Methods

1. Let us consider the following sigmoid

$$G(t) = \frac{1}{e-1} (e^{f(t)} - 1) \quad (4)$$

with

$$G(t_0) = \frac{1}{2}, \quad t_0 = \frac{1}{k} \ln \frac{1}{\frac{1}{e+1} - 1} \quad (5)$$

based on (1) with the Verhulst logistic function  $f(t)$ .

The one-sided H-distance  $d = \rho(h_{t_0}, G)$  between the shifted Heaviside step function  $h_{t_0}$  and the sigmoidal function  $G$  satisfies the relation:

$$G(t_0 + d) = \frac{1}{e-1} (e^{f(t_0+d)} - 1) = 1 - d. \quad (6)$$

The following theorem gives upper and lower bounds for  $d = d(k)$

**Theorem 2.1** The one-sided H-distance  $d(k)$  between the function  $h_{t_0}$  and the function  $G$  can be expressed in terms of the rate parameter  $k$  for any real  $k \geq 2$  as follows:

$$d_l = \frac{1}{2.5(1+0.254884k)} < d < \frac{\ln 2.5(1+0.254884k)}{2.5(1+0.254884k)} = d_r. \quad (7)$$

**Proof.** We define the functions

$$F_1(d) = \frac{1}{e-1} (e^{f(t_0+d)} - 1) - 1 + d \quad (8)$$

$$G_1(d) = -\frac{1}{2} + (1 + 0.254884k)d. \quad (9)$$

From Taylor expansion

$$\begin{aligned} \frac{1}{e-1} (e^{f(t_0+d)} - 1) - 1 + d - \left(-\frac{1}{2} + (1 + 0.254884k)d\right) \\ = O(d^2) \end{aligned}$$

we see that the function  $G_1(d)$  approximates  $F_1(d)$  with  $d \rightarrow 0$  as  $O(d^2)$  (see Figure 1).

In addition  $G'_1(d) > 0$  and for  $k \geq 2$

$$G_1(d_l) < 0; \quad G_1(d_r) > 0.$$

This completes the proof of the inequalities (7).

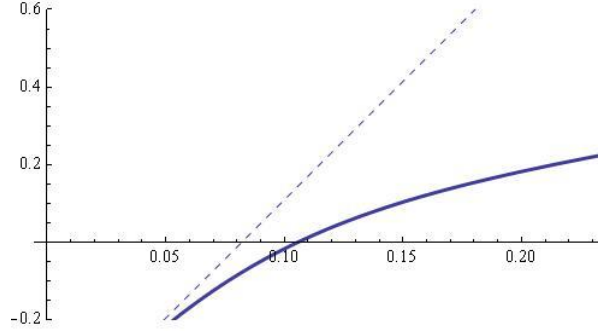


Fig. 1 - The functions  $F_1(d)$  and  $G_1(d)$  for  $k = 20$ .

The generated sigmoidal function  $G(t)$  for  $k = 20$  is visualized on Figure 2

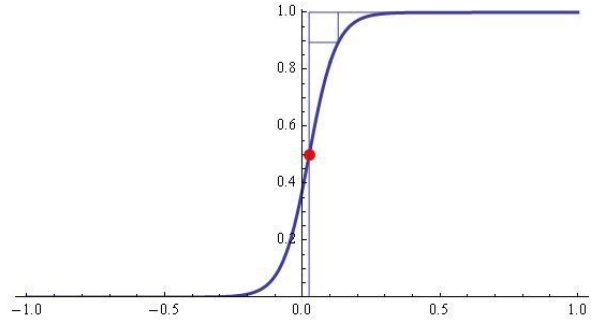


Fig. 2 - The H-distance  $d(k)$  between the functions  $h_{t_0}$  and  $G$  for  $k = 20$  is  $d = 0.105561$ ;  $d_l = 0.0655987$ ;  $d_r = 0.178704$ .

Some computational examples using relations (7) are presented in Table 1. The third column of Table 1 contains the value of  $d$  for prescribed values of  $k$  computed by solving the nonlinear equation (6).

Table 1. Bounds for  $d(k)$  computed by (6) and (7) for various rates  $k$

$k$	$d_l$	$d$ computed by (6)	$d_r$
30	0.0462014	0.0747728	0.142182
40	0.0357291	0.0627923	0.119042
50	0.0291032	0.0541761	0.102935
100	0.015101	0.0296749	0.0633183
500	0.00311425	0.00617859	0.0179747
1000	0.00156321	0.0048109	0.0100999

**Definition 5** Another popular transformation by using a (cdf)  $F(t)$  is (Kumar, Singh & Singh, 2017):

$$G1(t) = e^{1 - \frac{1}{F(t)}}. \quad (10)$$

2. Let us consider the following sigmoid

$$G1(t) = e^{1 - \frac{1}{f(t)}} \quad (11)$$

with

$$G1(t_0) = \frac{1}{2}, \quad t_0 = -\frac{1}{k} \ln(\ln 2) \quad (12)$$

based on (10) with the Verhulst logistic function  $f(t)$ .

The one-sided H-distance  $d_1 = \rho(h_{t_0}, G1)$  between the shifted Heaviside step function  $h_{t_0}$  and the sigmoidal function  $G1$  satisfies the relation:

$$G1(t_0 + d_1) = e^{1 - \frac{1}{f(t_0 + d_1)}} = 1 - d_1. \quad (13)$$

The following theorem gives upper and lower bounds for  $d_1 = d_1(k)$

**Theorem 2.2** *The one-sided H-distance  $d_1(k)$  between the function  $h_{t_0}$  and the function  $G1$  can be expressed in terms of the rate parameter  $k$  for any real  $k \geq 2$  as follows:*

$$d_{l_1} = \frac{1}{2.5(1+0.346574k)} < d_1 < \frac{\ln 2.5(1+0.346574k)}{2.5(1+0.346574k)} = d_{r_1}. \quad (14)$$

The proof follows the ideas given in this paper and will be omitted.

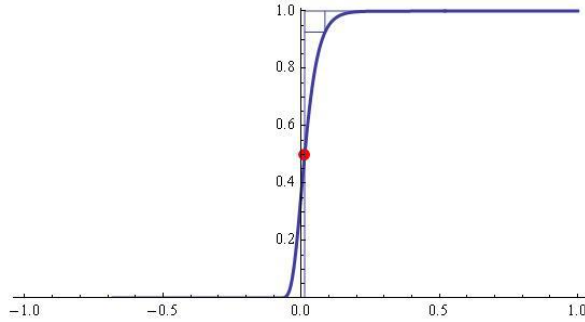


Fig. 3 - The H-distance  $d_1(k)$  between the functions  $h_{t_0}$  and  $G1$  for  $k = 30$  is  $d_1 = 0.0735233$ .

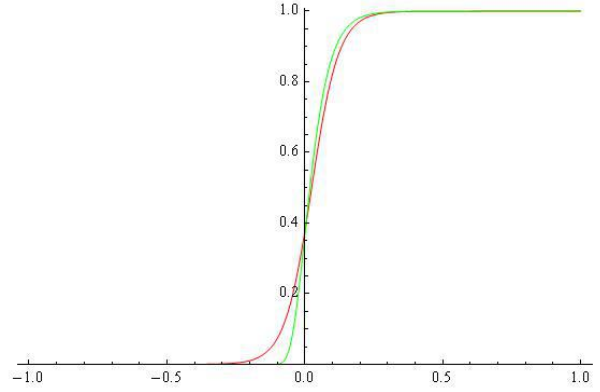


Fig. 4 - Comparison between  $G$  (red) and  $G1$  (green) for  $k = 20$ .

### 3 Results

To achieve our goal, we obtain new estimates for the one-sided H-distance between a shifted Heaviside step function and its best approximating family of transmuted cumulative distribution functions—these are the families of functions  $G(t)$  and  $G1(t)$  based on the Verhulst logistic function.

Numerical examples, illustrating our results are given.

In some cases the approximation of shifted Heaviside function by  $G1(t)$  is better in comparison to its approximation by  $G(t)$  (see Figure 4).

For other results, see (Iliev, Kyurkchiev & Markov, 2017a; Kyurkchiev, 2015; Kyurkchiev & Iliev, 2016; Kyurkchiev, V. & Kyurkchiev, N., 2015; Kyurkchiev & Markov, 2016c; Iliev, Kyurkchiev & Markov, 2017b; Kyurkchiev, V. & Kyurkchiev N., 2017; Kyurkchiev, 2016b).

```
Manipulate[Dynamic@Show[Plot[G[t], {t, -3, 3}, LabelStyle -> Directive[Green, Bold],
PlotLabel -> 1/(Exp[1] - 1) * (Exp[1/(1 + Exp[-k * t])] - 1)],
PlotRange -> {Automatic, {0, 1}}, AxesOrigin -> {0, 0}], {{k, 1}, 1, 100, Appearance -> "Open"},
{{t0, 0}, 0.01, 10, Appearance -> "Open"},
Initialization -> {G[t_] := 1/(Exp[1] - 1) * (Exp[1/(1 + Exp[-k * t])] - 1)}]
```

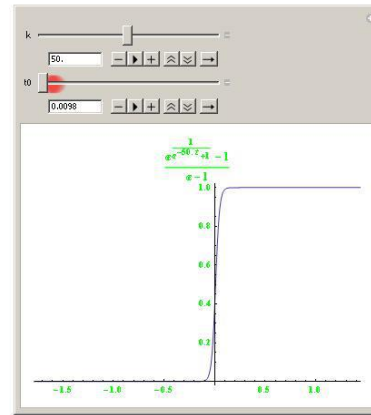


Fig. 5 - Software tools in CAS Mathematica.

We propose a software module within the programming environment *CAS Mathematica* for the analysis of the considered families of transmuted cumulative distribution functions.

The module offers the following possibilities:

- generation of the functions  $G(t)$  and  $G1(t)$  under user defined values of the reaction rate  $k$  and  $t_0$ ;
- calculation of the H-distance between the Heaviside function  $h_{t_0}$  and the sigmoidal functions  $G(t)$  and  $G1(t)$ ;
- software tools for animation and visualization.

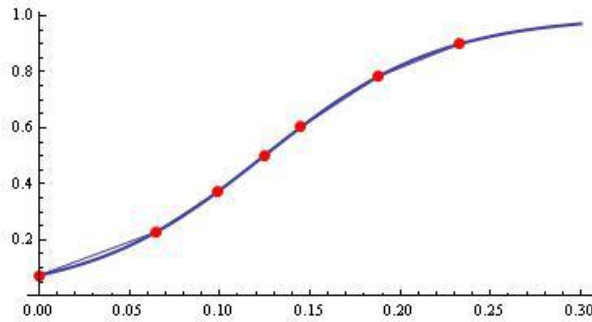
#### 4 Appendix

Focusing on the shifted logistic function

$$f_r(t) = \frac{1}{1 + e^{-k(t-r)}} \quad (15)$$

and the shifted function

$$G_r(t) = \frac{1}{e-1} \left( e^{f_r(t)} - 1 \right). \quad (16)$$



**Fig. 6 - Interpolation of the experimental data by model (16).**

We examine the following experimental data (biomass) for *Xantobacter autotrophicum* by the model  $G_r(t)$ .

Table 2. The experimental data (biomass) for *Xantobacter autotrophicum* and approximation by  $G_r(t)$  for  $k = 20$  and  $r = 0.1$

$t$	Biomass	$G_r(t)$
0	0.104	0.0736774
0.065	0.233	0.229003
0.099	0.39	0.372755
0.125	0.507	0.50254
0.145	0.618	0.602883
0.188	0.766	0.784021
0.233	0.88	0.899886

From Figure 6 it can be seen that the results are satisfactory. We point out that in similar "exponential" data type the results are near to Gompertz growth model.

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## ПРИМЕНЕНИЕ НЕКОТОРЫХ НОВЫХ ТРАНСМУТИРОВАННЫХ КУМУЛЯТИВНЫХ ФУНКЦИЙ РАСПРЕДЕЛЕНИЯ В ДИНАМИКЕ ПОПУЛЯЦИЙ

### Финансирование

Данная работа была выполнена при поддержке гранта FP17-FMI-008 Отдела Научных Исследований, Пловдивский Университет «Паисий Хилендарски»

### Конфликт интересов

Не указан.

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### Аннотация

**Мотивация:** В литературе представлено несколько преобразований для получения новой кумулятивной функции распределения (cdf) с помощью другой(-их) известной(-ых) cdf.

**Результаты:** В данной работе найдено применение некоторым новым преобразованиям кумулятивной функции распределения для построения семейства сигмоидальных функций на основе логистической функции Ферхюльста.

Посредством этого семейства мы доказываем предположения для хаусдорфовой аппроксимации сдвинутой ступенчатой функции Хевисайда. Приводятся численные примеры, иллюстрирующие полученные результаты.

**Ключевые слова:** Кумулятивная функция распределения, логистическая функция, сдвинутая ступенчатая функция Хевисайда, хайсдорфово расстояние, верхний и нижний пределы.